

EXPLICIT POLYNOMIALS HAVING THE HIGMAN-SIMS GROUP AS GALOIS GROUP OVER $\mathbb{Q}(t)$

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ABSTRACT. We compute explicit polynomials having the sporadic Higman-Sims group HS and its automorphism group $\text{Aut}(\text{HS})$ as Galois groups over the rational function field $\mathbb{Q}(t)$.

1. INTRODUCTION

From a theoretical perspective it is known that $\text{Aut}(\text{HS})$, the automorphism group of the sporadic Higman-Sims group HS, occurs as a Galois group over $\mathbb{Q}(t)$ since it has a rigid rational generating triple, see [2] and [5].

In order to find explicit polynomials with Galois group $\text{Aut}(\text{HS})$ over $\mathbb{Q}(t)$ one can compute a three-point branched covering, also called *Belyi map*, over $\mathbb{P}^1\mathbb{C}$ whose ramification corresponds to these rigid rational triples.

For a thorough survey on computing Belyi maps refer to [6]. Recently, Klug et al. calculated a Belyi map of degree 50 with monodromy group isomorphic to $\text{PSU}_3(\mathbb{F}_5)$ using modular forms, see [3].

We developed another efficient method of computing certain Belyi maps of higher degree which we will explain in detail in an upcoming paper. The purpose of the current note is to present a Belyi map of degree 100 with monodromy group isomorphic to $\text{Aut}(\text{HS})$. As a consequence, we obtain polynomials having HS and $\text{Aut}(\text{HS})$ as Galois groups over $\mathbb{Q}(t)$.

2. RAMIFICATION DATA AND COMPUTED RESULTS

Our goal is to compute a Belyi map $f : \mathbb{P}^1\mathbb{C} \rightarrow \mathbb{P}^1\mathbb{C}$ of ramification type $(x, y, z) \in S_{100}^3$ given by

$$\begin{aligned} x = & (1, 64, 8, 54, 37)(2, 20, 81, 42, 49)(3, 98, 32, 73, 89)(4, 96, 86, 15, 79) \\ & (5, 22, 28, 78, 48)(6, 67, 97, 40, 14)(7, 58, 82, 59, 18)(9, 16, 87, 85, 60) \\ & (10, 70, 41, 56, 55)(11, 77, 36, 25, 68)(12, 17, 19, 21, 80)(13, 35, 90, 33, 91) \\ & (23, 50, 66, 84, 27)(24, 72, 95, 52, 76)(26, 99, 100, 57, 93)(29, 71, 38, 69, 65) \\ & (30, 74, 94, 53, 51)(31, 45, 47, 75, 34)(43, 63, 44, 46, 62), \end{aligned}$$

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$$\begin{aligned}
y &= (1, 20)(2, 64)(3, 76)(4, 45)(5, 83)(6, 26)(7, 13)(8, 74)(9, 41)(10, 63)(11, 25) \\
&\quad (12, 66)(14, 21)(15, 52)(16, 62)(17, 33)(18, 35)(19, 42)(22, 60)(23, 58) \\
&\quad (24, 73)(28, 98)(29, 82)(30, 53)(31, 61)(32, 59)(34, 67)(36, 95)(37, 85) \\
&\quad (38, 47)(39, 51)(40, 80)(43, 92)(44, 78)(46, 99)(48, 55)(49, 94)(50, 91) \\
&\quad (54, 90)(65, 88)(69, 72)(71, 75)(77, 79)(81, 87)(84, 97)(86, 100)(93, 96), \\
z &= (1, 2)(13, 18)(21, 40)(47, 71)(25, 68)(15, 95, 77)(20, 37, 87)(39, 53, 51) \\
&\quad (49, 74, 64)(55, 78, 63)(57, 100, 96)(66, 80, 97)(73, 76, 89)(7, 91, 23) \\
&\quad (12, 50, 33)(22, 85, 54, 35, 59, 98)(24, 32, 82, 65, 88, 69) \\
&\quad (3, 52, 86, 99, 44, 28)(4, 31, 61, 34, 6, 93)(5, 83, 48, 56, 41, 60) \\
&\quad (8, 30, 94, 42, 17, 90)(9, 70, 10, 43, 92, 62)(11, 36, 72, 38, 45, 79) \\
&\quad (14, 19, 81, 16, 46, 26)(27, 84, 67, 75, 29, 58).
\end{aligned}$$

This permutation triple is of the following type:

	x	y	z
cycle structure	$5^{19}.1^5$	$2^{47}.1^6$	$6^{10}.3^{10}.2^5$

With the help of **Magma** [1] we can easily verify:

- $x \cdot y \cdot z = 1$
- $\text{Aut}(\text{HS}) = \langle x, y \rangle$
- (x, y, z) is a rigid and rational triple of genus 0

Due to the rational rigidity criterion [7, p. 48] and a rationality consideration there exists a Belyi map $f \in \mathbb{Q}(X)$ of degree 100 with monodromy group isomorphic to $\text{Aut}(\text{HS})$. Note that f is unique, up to inner and outer Möbius transformations.

Applying our newly developed method we were able to compute this Belyi map explicitly. The resulting function $f : \mathbb{P}^1\mathbb{C} \rightarrow \mathbb{P}^1\mathbb{C}$ is of the form

$$f(X) = \frac{p(X)}{q(X)} = 1 + \frac{r(X)}{q(X)}$$

where

$$\begin{aligned}
p(X) &= 3^3 \cdot (X^4 - 8X^3 - 6X^2 + 8X + 1)^5 \cdot \\
&\quad (X^5 - 5X^4 + 50X^3 + 70X^2 + 25X + 3)^5 \cdot \\
&\quad (3X^5 - 5X^4 - 5X^3 + 35X^2 + 40X + 4) \cdot \\
&\quad (9X^{10} - 30X^9 + 55X^8 - 200X^7 + 210X^6 + 924X^5 \\
&\quad - 890X^4 - 360X^3 + 1925X^2 - 1070X + 291)^5, \\
q(X) &= (3X^5 - 35X^4 + 90X^3 - 50X^2 + 15X + 9)^2 \cdot \\
&\quad (9X^{10} - 120X^9 + 10X^8 - 1960X^7 - 1090X^6 + 3304X^5 \\
&\quad - 760X^4 - 920X^3 + 145X^2 + 80X + 6)^3 \cdot \\
&\quad (3X^{10} - 10X^9 - 65X^8 + 160X^7 - 90X^6 - 932X^5 \\
&\quad - 330X^4 + 880X^3 + 1255X^2 + 830X + 27)^6
\end{aligned}$$

and

$$\begin{aligned} r(X) &= p(X) - q(X) \\ &= 2^2 \cdot 3^{14} \cdot 5^3 \cdot (X-1) \cdot r_5(X) \cdot r_{10}^2(X) \cdot r_{16}^2(X) \cdot r_{20}^2(X) \end{aligned}$$

with irreducible monic polynomials r_j of degree j .

From the factorizations of p , q , r and the Riemann-Hurwitz formula it is clear that f is indeed a three-point branched cover of $\mathbb{P}^1\mathbb{C}$, ramified over $0, 1$ and ∞ .

3. VERIFICATION OF MONODROMY

We will present two proofs to verify that the monodromy group of our Belyi map $f = p/q$ is isomorphic to $\text{Aut}(\text{HS})$.

First, one can compute the corresponding dessin d'enfant, i.e. the bipartite graph drawn on the Riemann sphere $\mathbb{P}^1\mathbb{C}$ obtained by taking the elements of $f^{-1}(0)$ as black vertices, those of $f^{-1}(1)$ as white vertices and the connected components of $f^{-1}((0, 1))$ as edges, labelled from 1 to 100. A part of this bipartite graph is shown in Figure 1. Note that the poles of f are marked by 'x'. Listing the cyclic arrangement of adjacent edges around each black and white vertex, respectively, we obtain the cycles of x and y , up to simultaneous conjugation. Thus the monodromy group of f is isomorphic to $\text{Aut}(\text{HS})$.

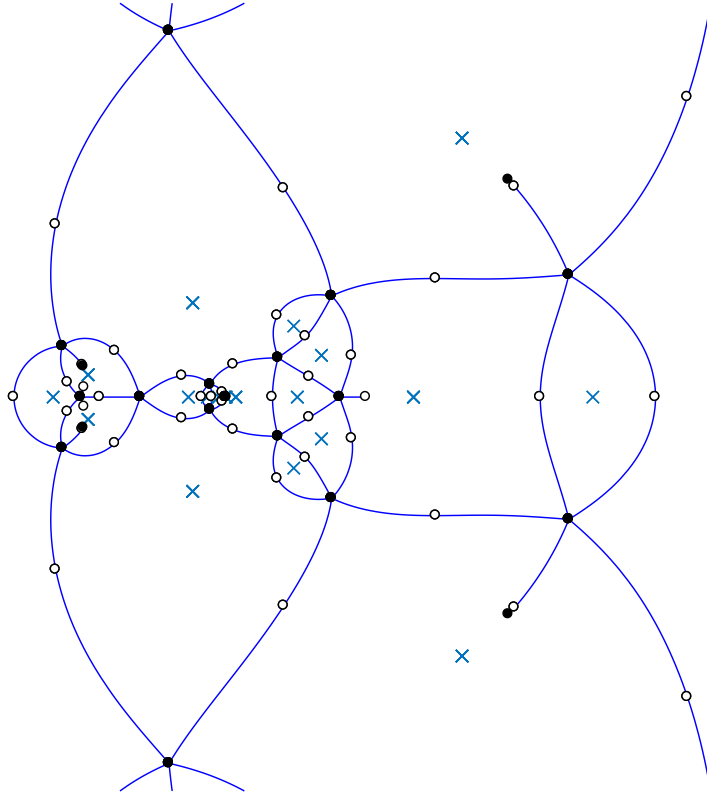


FIGURE 1. Dessin d'enfant corresponding to f

Another way to verify the monodromy can be done algebraically: The monodromy group of f can be viewed as the Galois group $\text{Gal}(p(X) - tq(X) \mid \mathbb{Q}(t))$ or equivalently $\text{Gal}(p(X) - f(t)q(X) \mid \mathbb{Q}(f(t)))$.

First note that $\text{Gal}(p(X) - f(t)q(X) \mid \mathbb{Q}(t))$ equals the point stabilizer of t in the permutation group $\text{Gal}(p(X) - f(t)q(X) \mid \mathbb{Q}(f(t)))$ acting transitively on the 100 roots of $p(X) - f(t)q(X)$.

As $p(X) - f(t)q(X)$ factorizes over $\mathbb{Q}(t)[X]$ into three irreducible polynomials of degrees 1, 22 and 77, respectively, we see that $\text{Gal}(p(X) - f(t)q(X) \mid \mathbb{Q}(f(t)))$ and thus also $G := \text{Gal}(p(X) - tq(X) \mid \mathbb{Q}(t))$ are rank 3 permutation groups of degree 100 with subdegrees 1, 22 and 77.

We now show that G is actually a primitive permutation group: Suppose $G \leq \text{Sym}(\Omega)$, $|\Omega| = 100$, has some non-trivial block Δ , i.e. $1 < |\Delta| < 100$, such that for each $g \in G$ either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. Now fix some $\omega \in \Delta$. Then the stabilizer G_ω must leave Δ invariant, and — as G is rank 3 group — G_ω has exactly the non-empty orbits $\{\omega\}$, $\Delta \setminus \{\omega\}$ and $\Omega \setminus \Delta$. Knowing the sizes of the suborbits we find that Δ has either length $1 + 22 = 23$ or length $1 + 77 = 78$. This is a contradiction as the size of a block must always divide the permutation degree, in our case 100.

Now, combining the classification of all finite primitive rank 3 permutation groups (see e.g. [4]) with the subdegrees of G , only two possibilities remain: $G = \text{Aut}(\text{HS})$ or $G = \text{HS}$.

Since HS, in contrary to $\text{Aut}(\text{HS})$, is an even permutation group, it suffices to check whether the discriminant δ of $p(X) - tq(X) \in \mathbb{Q}(t)[X]$ is a square in $\mathbb{Q}(t)$. Using **Magma** we see $\delta = u^2 2(t - 1)$ for some $u \in \mathbb{Q}(t)$ and therefore $G = \text{Aut}(\text{HS})$.

Remark. By applying the previous arguments to $p(X) - (2t^2 + 1)q(X)$ we find $\text{Gal}(p(X) - (2t^2 + 1)q(X) \mid \mathbb{Q}(t))$ is either HS or $\text{Aut}(\text{HS})$. The discriminant, however, is a square now, thus $\text{Gal}(p(X) - (2t^2 + 1)q(X) \mid \mathbb{Q}(t)) = \text{HS}$.

4. ANOTHER EXAMPLE

Essentially, $\text{Aut}(\text{HS})$ contains exactly two rigid rational generating triples of genus 0. The first one has been discussed in the previous section. The second triple $(x, y, z) \in S_{100}$ where

$$\begin{aligned} x = & (1, 23, 53, 86)(2, 36, 29, 43)(3, 15, 46, 6)(4, 80, 71, 81)(5, 75, 16, 47) \\ & (7, 32, 60, 8)(9, 76, 100, 51)(10, 50, 49, 34)(11, 28, 74, 84)(12, 72, 37, 52) \\ & (13, 21, 96, 88)(14, 41, 40, 87)(17, 42, 45, 79)(18, 63, 19, 20)(22, 99, 39, 89) \\ & (24, 59, 77, 38)(25, 68, 26, 35)(27, 69, 73, 48)(30, 92, 33, 82)(31, 56, 93, 58) \\ & (44, 98, 67, 64)(54, 95, 85, 62)(55, 65, 94, 61)(57, 78, 83, 97)(66, 90, 70, 91), \end{aligned}$$

$$\begin{aligned} y = & (1, 75, 5, 71, 15)(2, 43, 52, 89, 39)(3, 18, 100, 33, 35, 26, 58, 32, 53, 23) \\ & (4, 81, 47, 16, 86, 7, 42, 38, 77, 59)(6, 41, 14, 87, 82, 76, 9, 97, 19, 63) \\ & (8, 60, 93, 56, 13, 61, 36, 99, 70, 45)(10, 65, 55, 88, 12, 29, 94, 34, 49, 50) \\ & (11, 44, 64, 25, 92)(17, 72, 96, 69, 28, 30, 40, 46, 80, 24)(20, 83, 78, 57, 51) \\ & (21, 31, 68, 67, 98, 84, 74, 27, 48, 73)(22, 37, 79, 90, 66, 95, 54, 62, 85, 91) \end{aligned}$$

and $z := (xy)^{-1}$

of ramification type

	x	y	z
cycle structure	4^{25}	$10^8.5^4$	$2^{35}.1^{30}$

leads to the Belyi map

$$f(X) = \frac{p(X)}{q(X)} = 1 + \frac{r(X)}{q(X)}$$

where

$$\begin{aligned} p(X) &= (7X^5 - 30X^4 + 30X^3 + 40X^2 - 95X + 50)^4 \cdot \\ &\quad (2X^{10} - 20X^9 + 90X^8 - 240X^7 + 435X^6 - 550X^5 \\ &\quad + 425X^4 - 100X^3 - 175X^2 + 250X - 125)^4 \cdot \\ &\quad (2X^{10} + 5X^8 - 40X^6 + 50X^4 - 50X^2 + 125)^4, \\ q(X) &= 2^8 \cdot (X^4 - 5)^5 \cdot \\ &\quad (X^8 - 20X^6 + 60X^5 - 70X^4 + 100X^2 - 100X + 25)^{10}. \end{aligned}$$

Of course, it remains to verify that this rational function is indeed a three-point branched cover having the desired monodromy group. However, this can be done in the exact same way we already demonstrated in the previous section.

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